Journal of Statistical Physics, Vol. 75, Nos. 5/6, 1994

Random Shearing Direction Models for Isotropic Turbulent Diffusion

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Received November 18, 1993

Recently, a rigorous renormalization theory for various scalar statistics has been developed for special modes of random advection diffusion involving random shear layer velocity fields with long-range spatiotemporal correlations. New random shearing direction models for isotropic turbulent diffusion are introduced here. In these models the velocity field has the spatial second-order statistics of an arbitrary prescribed stationary incompressible isotropic random field including long-range spatial correlations with infrared divergence, but the temporal correlations have finite range. The explicit theory of renormalization for the mean and second-order statistics is developed here. With ε the spectral parameter, for $-\infty < \varepsilon < 4$ and measuring the strength of the infrared divergence of the spatial spectrum, the scalar mean statistics rigorously exhibit a phase transition from mean-field behavior for $\varepsilon < 2$ to anomalous behavior for ε with $2 < \varepsilon < 4$ as conjectured earlier by Avellaneda and the author. The universal inertial range renormalization for the second-order scalar statistics exhibits a phase transition from a covariance with a Gaussian functional form for ε with $\varepsilon < 2$ to an explicit family with a non-Gaussian covariance for ε with $2 < \varepsilon < 4$. These non-Gaussian distributions have tails that are broader than Gaussian as ε varies with $2 < \varepsilon < 4$ and behave for large values like $\exp(-C_\varepsilon |x|^{4-\varepsilon})$, with C_ε an explicit constant. Also, here the attractive general principle is formulated and proved that every steady, stationary, zero-mean, isotropic, incompressible Gaussian random velocity field is well approximated by a suitable superposition of random shear layers.

KEY WORDS: Turbulent diffusion; shear layers; long-range correlations; renormalization.

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0022-4715/94/0600-1153$07.00/0

1. INTRODUCTION

The statistical behavior of a passive scalar T satisfying the advection diffusion equation

$$\frac{\partial T}{\partial t}+(\bar{w}+v)\cdot\nabla T=\kappa\,\Delta T,\qquad x\in R^d,\quad t>0 \tag{1}$$

when the velocity field v is incompressible, i.e., div $v=0$, and known only statistically, is a well-known important practical and theoretical problem for turbulent fluid flows,[1,2] tracers in porous media,[3] etc. The constant vector $\bar{w}$ in (1) represents the mean flow. The problems in (1) are especially difficult when the velocity field involves a wide range of spatiotemporal scales, and the standard approaches in the physics community involve renormalized perturbation theories developed according to various formal recipes.[1-3]

Recently, a rigorous theory of renormalization for various scalar statistics has been developed for special models involving simple shear layers with turbulent velocity fields having long-range spatiotemporal correlations.[4-8] These models are the special case of (1) with the form

$$\frac{\partial T}{\partial t}+(\bar{w}+v_\omega(x\cdot\omega,t))\cdot\nabla T=\kappa\,\Delta T \tag{2}$$

Here ω is an arbitrary fixed direction on the unit sphere S^{d-1}, $\bar{w}$ is a constant vector representing the mean flow, and v_ω is a shear layer velocity field, i.e.,

$$v_\omega=P(\omega)\,\tilde{v}(x\cdot\omega,t) \tag{3}$$

with $P(\omega)$ the matrix projection, $P(\omega)=I-\omega\otimes\omega$, and $\tilde{v}={}^t(\tilde{v}_1(\tilde{x},t),...,\tilde{v}_d(\tilde{x},t))$ a vector of random velocity fields in $1+1$ dimensions with suitable long-range correlations. The matrix projection $P(\omega)$ guarantees that the velocity field $v_\omega(x\cdot\omega,t)$ is incompressible and consists of random plane waves involving shear flows. For the special case with $d=2$, there is a simplified representation for v_ω given by

$$v_\omega(x,t)=\omega^\perp\tilde{v}(x\cdot\omega,t) \tag{4}$$

where $\tilde{v}(\tilde{x},t)$ is a single scalar random field $\omega^\perp=(-\omega_2,\omega_1)$.

The purpose of this paper is first to introduce random shearing direction models for isotropic turbulent dciffusion. In these models the velocity field in (1) has the spatial statistics of an arbitrary stationary, incompressible, isotropic Gaussian random field including long-range correlations with infrared divergence, but the temporal correlations have finite range. The renormalization theory for both the mean statistics $\langle T \rangle$ and the second-order statistics $\langle T(x+x', t)\, T(x', t) \rangle$ is developed here following refs. 4–6 for the cases with simple shear layers. Here and below, the quantity $\langle \cdot \rangle$ denotes averaging over the random velocity statistics.

The formulas for the renormalization theory for the mean statistics developed in Section 3 rigorously confirm the "phase diagrams" for eddy diffusivity conjectured in ref. 9 for $z = 0$, $-\infty < \varepsilon < 4$. Here ε is the spectral parameter measuring the strength of the infrared divergence[4,6,9] (see Section 3). With the special assumptions on the velocity spectrum imposed for simplicity in Section 4, the universal inertial range renormalization for the second-order scalar statistics exhibits a phase transition from a covariance with a Gaussian functional form for ε with $\varepsilon < 2$ to an explicit family with a non-Gaussian functional form for ε with $2 < \varepsilon < 4$. These universal non-Gaussian distributions always have tails that are broader than Gaussian as ε varies with $2 < \varepsilon < 4$ and behave like $\exp(-C_\varepsilon |x|^{4-\varepsilon})$ with C_ε an explicit constant. Also, through similar considerations, the attractive general principle is formulated and proved that every steady, stationary, zero-mean, isotropic, incompressible, Gaussian random velocity field is well approximated by a suitable superposition of random shear layers; these ideas form the theoretical basis for attractive new Monte Carlo algorithms for turbulent diffusion with many spatial scales.[10,11] A complete rigorous renormalization theory for the higher-order scalar statistics in the isotropic models exhibiting interesting intermittency corrections is being developed for these isotropic models along the lines of ref. 6 for the special case of simple shear layers. These and related ideas will be presented elsewhere in other, more lengthy publications by the author.[12–14]

2. THE BASIC RANDOM SHEAR DIRECTION MODELS

To define the basic random shearing direction models, I consider a collection of independent identical random velocity fields $\{v^j(\tilde{x}, t)\}_{j=1}^{\infty}$ in $1+1$ dimensions where for each j, the components v_i^j for $1 \leqslant i \leqslant d$ are independent, zero-mean, stationary Gaussian random fields, completely characterized by the correlation function

$$R(\tilde{x}, t) = \langle v_i^j(x' + \tilde{x}, t + t')\, v_i^j(x', t') \rangle = f(t) \int_0^\infty \cos(2\pi \tilde{x} \tilde{k})\, E(\tilde{k})\, d\tilde{k} \quad (5)$$

Here $E(\tilde{k})$ is twice the real energy spectrum, $f(t)$ is a temporal correlation structure function, and, by independence, $\langle v_{i_1}^{j_1} v_{i_2}^{j_2} \rangle = 0$ for either $i_1 \neq i_2$ or

$j_1 \neq j_2$. I consider the following two special cases here for simplicity in exposition:

$$\begin{aligned} &\text{Model A:} \quad f(t) = \delta(t), \qquad \text{white noise decorrelation} \\ &\text{Model B:} \quad f(t) \equiv 1, \qquad \text{steady velocity fields} \end{aligned} \tag{6}$$

Pick a collection of independent random variables $\{\omega_j\}$ which are unit directions uniformly distributed on the unit sphere S^{d-1}. With a given correlation time $\bar{t}_c$, define an incompressible Gaussian random field $v^{\{\omega_j\}}$ in R^d by

$$v^{\{\omega_j\}}(x, t) = P(\omega_j)\, \tilde{v}^j(x \cdot \omega_j, t) \qquad \text{for} \quad (j-1)\, \bar{t}_c \leqslant t < j\bar{t}_c \tag{7}$$

The basic random shearing direction models which are discussed here involve the advection–diffusion equation in (1) *with the random velocity field* $v^{\{\omega_j\}}$ *in* (7).

Next, I show that *the random shear direction velocity fields can be constructed to yield the same spatial second-order statistics as a arbitrary prescribed stationary, zero-mean, incompressible random velocity field through an appropriate choice of* $E(\tilde{k})$ *in* (5). Recall[15] that the second-order correlation matrix of an incompressible stationary random field is given by

$$\begin{aligned} \langle v(x + x') \otimes v(x') \rangle &= \int_{R^d} \cos(2\pi x \cdot k) \frac{E(|k|)\, |k|^{1-d}}{A_d} P\left(\frac{k}{|k|}\right) dk \\ &= \fint_{S^{d-1}} \int_0^\infty \cos(2\pi\omega \cdot x\tilde{k})\, E(\tilde{k})\, P(\tilde{k})\, d\tilde{k}\, d\omega \end{aligned} \tag{8}$$

with A_d the area of S^{d-1} and $E(|k|)$ a prescribed energy density; in the last equality in (8), polar coordinates have been used and $\fint_{S^{d-1}}$ denotes the normalized integral over the unit sphere. Next, I compute the equal-time second-order correlations for the random shearing direction velocity field from (7),

$$\begin{aligned} &\langle v^{\{\omega_j\}}(x + x', t) \otimes v^{\{\omega_j\}}(x', t) \rangle \\ &= \fint_{S^{d-1}} \langle P(\omega)\, \tilde{v}^j((x + x') \cdot \omega, t) \otimes P(\omega)\, \tilde{v}^j(x' \cdot \omega, t) \rangle\, d\omega \\ &= \fint_{S^{d-1}} \left(\sum_{i=1}^{d} P(\omega)\, e_i \otimes P(\omega)\, e_i \right) \int_0^\infty \cos(2\pi x \cdot \omega\tilde{k})\, E(\tilde{k})\, d\tilde{k}\, d\omega \end{aligned} \tag{9}$$

where (5) has been used in the last equality and $\{e_i\}_{i=1}^d$ is the standard basis in R^d. I claim that the following matrix identity is valid:

$$\sum_{i=1}^{d} P(\omega)\, e_i \otimes P(\omega)\, e_i = P(\omega) \tag{10}$$

By inserting the identity in (10) in the last equality in (9), one obtains the identical formula as given in the last equality in (8). The identity in (10) is an elementary exercise for the reader utilizing the fact that $P(\omega)$ is a matrix projection so that $P(\omega)$ is symmetric and $P^2(\omega) = P(\omega)$.

2.1. An Important Variation

I claim that an *arbitrary steady, stationary, zero-mean, isotropic, incompressible Gaussian random velocity field* $v(x)$ *is well-approximated by a superposition of shear layers.* With $\{\tilde{v}^j(x)\}_{j=1}^{\infty}$ the independent random steady velocity fields from (5) for model B, consider the *stationary, zero-mean, incompressible Gaussian random field* $v_N^{\{\omega_j\}}$ formed by a spatial superposition of shear layers and given by

$$v_N^{\{\omega_j\}}(x) = \sum_{j=1}^{N} N^{-1/2} P(\omega_j)\, \tilde{v}^j(x \cdot \omega_j) \tag{11}$$

where ω_j are arbitrary unit directions in S^{d-1}. By repeating the calculations in (8) and (9) without averaging over $\{\omega_j\}$, it is a simple matter to calculate that the second-order correlation matrices satisfy

$$\begin{aligned} &\langle v_N^{\{\omega_j\}}(x+x') \otimes v_N^{\{\omega_j\}}(x') \rangle - \langle v(x+x') \otimes v(x') \rangle \\ &\quad = \frac{1}{N} \sum_{j=1}^{N} F(x, \omega_j) - \fint_{S^{d-1}} F(x, \omega)\, d\omega \end{aligned} \tag{12}$$

with $F(x, \omega) = R(x \cdot \omega)\, P(\omega)$ and $R(\tilde{x})$ from (5). If the unit directions ω_j are chosen at random from the uniform distribution on S^{d-1}, the right-hand side of (12) converges to zero in absolute value as N increases by the law of large numbers; thus the incompressible Gaussian random field defined in (11) as a finite superposition of random shear layers converges to the incompressible, isotropic Gaussian random field with the second-order correlation function in (8) because both are Gaussian random fields and the two-point correlation converge.

The fact elucidated in the previous paragraph yields a new numerical strategy for computing incompressible Gaussian random velocity fields with many spatial scales provided one has an efficient numerical algorithm for the simple shear layer with many spatial scales and a suitable variance reduction algorithm for the direction sampling in (12). Elliott and Majda devise such an algorithm with these attractive features for velocity fields with infrared divergent spectra in refs. 10 and 11. Furthermore, the method readily generalizes to compressible, isotropic Gaussian random velocity fields and there is an explicit convergence theory for the overall algorithm.[11] In two space dimensions the algorithm simplifies even further through the use of (4).

3. SCALAR MEAN STATISTICS

Here I consider the scalar mean statistics $\langle T(x+\bar{w}t, t)\rangle$ defined in a reference frame moving with the mean flow for the problem in (1) with the random shear direction velocity field defined in (7). For model A statistics with sharp decorrelation in time, the large-scale sweeping effect from $\bar{w}$ will be zero, but these effects are nontrivial for model B.

For the simple shear layer models in (2), the mean statistics are exactly solvable for $\kappa \geqslant 0$ with explicit formulas in refs. 6 and 7; with the random shearing model from (7), the mean statistics for the scalar are determined by averaging these formulas over S^{d-1}. The result is the explicit *Fourier representation formula* for the *scalar mean statistics for model A*: For $l\bar{t}_c \leqslant t < (l+1)\,\bar{t}_c$ and any $l = 1, 2,...,$

$$\langle T(x+\bar{w}t, t)\rangle = \int_{R^d} e^{2\pi i x\cdot k}(M_A(k, \bar{t}_c))^l\, M_A(k, t-l\bar{t}_c)\,\hat{T}_0(k)\,dk \tag{13}$$

with the Fourier multiplier $M_A(k, t)$ given by

$$M_A(k, t) = \exp(-4\pi^2\,|k|^2\,\kappa t) \fint_{S^{d-1}} \exp\left[-4\pi^2\,\frac{R(0)}{4}\,t\,|P(\omega)k|^2\right] d\omega \tag{14}$$

In (13), $\hat{T}_0(k)$ denotes the Fourier transform of the initial data. For model B with $\kappa = 0$, there is a similar exact representation formula[8] for the scalar mean statistics for the shear layer with $\kappa = 0$ and $\bar{w} \neq 0$; this yields an analogous formula to that in (13) with the Fourier multiplier $M_B(k, t, \bar{w})$ given for $d = 2$ by

$$M_B(k, t, \bar{w}) = \fint_{S^1} \exp[-4\pi^2\,|\omega^{\perp}\cdot k|^2\,\mathscr{D}(t, \omega, \bar{w})]\,d\omega \tag{15}$$

with

$$\mathscr{D}(t, \omega, \bar{w}) = \tfrac{1}{2}\int_0^t \int_0^t R((\bar{w}\cdot\omega^{\perp})(s-s'))\,ds\,ds' \tag{16}$$

3.1. Large-Scale Renormalization Theory

Following refs. 4 and 5, I consider the random shear direction velocity fields defined in (5) and (7) with a parametrized family of energy spectra given by

$$E_\delta^\varepsilon(k) = \begin{cases} \frac{1}{2} V^2\,|k|^{1-\varepsilon}\psi_\infty(|k|), & |k| \geqslant \delta \\ 0, & |k| < \delta \end{cases} \tag{17}$$

and the spectral parameter ε varying for $-\infty<\varepsilon<4$. Here $\psi_\infty(|k|)$ is a smooth, rapidly decreasing ultraviolet cutoff with $\psi_\infty(0)=1$. For ε with $2<\varepsilon<4$ there is infrared divergence of energy since $R_\delta^\varepsilon(0)$ from (5) satisfies $R_\delta^\varepsilon(0)\to\infty$ in this regime. Following refs. 4, 5, and 9, I consider the deterministic large-scale initial data $T_0(\delta x)$ with $\delta\ll 1$ and seek a nontrivial large-time rescaling function $\rho^2(\delta)$ so that the mean statistics

$$\bar{T}_\delta(x,t)=\left\langle T\left(\frac{x}{\delta}-\bar{w}\frac{t}{\rho^2},\frac{t}{\rho^2}\right)\right\rangle \tag{18}$$

satisfy a nontrivial equation in the limit $\delta\to 0$. This is the "eddy diffusivity" problem in the model.[4,5,9]

For model A it follows from (13) that

$$\bar{T}_\delta(x,t)=\int_{R^d} e^{2\pi i x\cdot k} M_{\rm A}(\delta k,\bar{t}_c)^{[t/(\rho^2\bar{t}_c)]} M_{\rm A}\left(\delta k,\frac{t}{\rho^2}-\bar{t}_c\left[\frac{t}{\rho^2\bar{t}_c}\right]\right)\hat{T}_0(k)\,dk \tag{19}$$

To find the time scaling function $\rho(\delta)$ and the limiting effective equation satisfied by $\bar{T}_{\rm A}(x,t)=\lim_{\delta\to 0}\bar{T}_\delta(x,t)$, I follow the standard Fourier analysis proof of the central limit theorem and calculate the behavior of

$$\exp\left[\ln(M_{\rm A}(\delta k,\bar{t}_c))\frac{t}{\rho^2\bar{t}_c}\right]$$

as δ, $\rho^2(\delta)\to 0$. With (14) one computes that

$$\ln(M_{\rm A}(\delta k,\bar{t}_c))\frac{t}{\rho^2\bar{t}_c}=-4\pi^2|k|^2 t\left[\frac{\delta^2}{\rho^2}\kappa+\frac{\delta^2}{\rho^2}c_d R_\delta^\varepsilon(0)\right]+o([\cdot]) \tag{20}$$

where $c_d=\frac{1}{2}\textstyle\fint_{S^{d-1}}|P(\omega)\,e_1|^2\,d\omega$.

3.1.1. The Mean-Field Regime: $-\infty<\epsilon<2$.

For ε with $\varepsilon<2$,

$$\lim_{\delta\to 0} R_\delta^\varepsilon(0)=R^\varepsilon(0)=V^2\int_0^\infty |k|^{1-\varepsilon}\psi_\infty(|k|)\,dk$$

Thus, with the usual diffusive scaling $\rho(\delta)=\delta$ and (19), (20), $\bar{T}_{\rm A}(x,t)=\lim_{\delta\to 0}\bar{T}_\delta(x,t)$ satisfies the renormalized diffusion equation

$$\frac{\partial\bar{T}_{\rm A}}{\partial t}=[\kappa+c_d R^\varepsilon(0)]\,\Delta\bar{T}_{\rm A},\qquad \bar{T}_{\rm A}|_{t=0}=T_0(x) \tag{21}$$

A similar set of computations with the explicit formulas in (15), (16) for model B velocity statistics with $\rho(\delta)=\delta$ yields the different renormalized diffusion equation for $\bar{T}_B(x, t)=\lim_{\delta\to 0}\bar{T}_\delta(x, t)$ given by

$$\frac{\partial \bar{T}_{\mathrm{B}}}{\partial t}=\sum_{i,j=1}^{2}\mathscr{D}_{\mathrm{B}}^{ij}\frac{\partial^2 \bar{T}_{\mathrm{B}}}{\partial x_i\,\partial x_j},\qquad \bar{T}_{\mathrm{B}}|_{t=0}=T_0(x) \tag{22}$$

with the matrix $\mathscr{D}_{\mathrm{B}}$ given by

$$\mathscr{D}_{\mathrm{B}}(\bar{w})=\frac{1}{\bar{t}_c}\int_{S^1}(\omega^\perp\otimes\omega^\perp)\,\mathscr{D}(\bar{t}_c, \omega, \bar{w})\,d\omega \tag{23}$$

with $\mathscr{D}(t, \omega, \bar{w})$ from (16). In the mean-field regime, in general $\mathscr{D}_{\mathrm{B}}(\bar{w})$ is an anisotropic matrix for a given large-scale mean flow[12] $\bar{w}$; thus, the sweeping effect in model B results in the different anisotropic effective limiting equation in (22) as compared with the isotropic equation in (21) for model A.

3.1.2. The Anomalous Regime: $2<\epsilon<4$. For ε with $2<\varepsilon<4$, where there is infrared divergence of energy, from (17) it follows than

$$R_\delta(0)=\delta^{2-\varepsilon}\left[\frac{V^2}{\varepsilon-2}+o(1)\right]\qquad \text{as}\quad \delta\to 0 \tag{24}$$

Furthermore, $R_\delta^\varepsilon(\tilde{x})$ satisfies $|R_\delta^\varepsilon(\tilde{x})-R_\delta^\varepsilon(0)|\leqslant C_1$ for $|\tilde{x}|<C_2$ in the regime $2<\varepsilon<4$. Thus, (20) and (24) yield the anomalous time rescaling $\rho(\delta)=\delta^{2-\varepsilon/2}$ for $2<\varepsilon<4$ with the effective equation

$$\frac{\partial \bar{T}}{\partial t}=D_{\mathrm{A,B}}\,\Delta\bar{T},\qquad \bar{T}|_{t=0}=T_0(x) \tag{25}$$

valid for velocity statistics in both models A and B with $D_{\mathrm{A}}=V^2/(\varepsilon-2)$ and $D_{\mathrm{B}}=\bar{t}_c D_{\mathrm{A}}$. In particular, the sweeping effect of the large scales $\bar{w}$ is negligible for the velocity statistics from model B in the anomalous regime in contrast with the mean-field regime.[5,9]

All of the results in this section provide an elementary but rigorous confirmation of the behavior of the renormalized scalar mean statistics for isotropic incompressible velocity fields conjectured in ref. 9.

4. INERTIAL RANGE BEHAVIOR FOR THE SECOND-ORDER STATISTICS

The random shearing direction models provide a convenient framework for rigorously justifying[13,14] various explicit equations for renormalized higher-order scalar statistics in the inertial range for isotropic turbulent diffusion in parallel to the program from ref. 6 for the simple shear layer case with long-range spatial correlations and white noise decorrelation in time. Here I briefly discuss this strategy without any details for the second-order scalar correlations and then briefly explore the implications for inertial range renormalization in a special case. I assume that the initial scalar distribution $T_0(x)$ is a stationary Gaussian random field[6] with a spectral density which is smooth and rapidly decreasing with the unit value at zero.

The goal here is to derive effective equations and universal scaling behavior for the second-order correlations $\langle T(x+x', t)\, T(x', t)\rangle = Q(x, t)$, where the average $\langle \cdot \rangle$ denotes ensemble average over both the random initial data and the velocity statistics. For model A velocity statistics and $\kappa \geqslant 0$[6,7] and model B velocity statistics with $\kappa = 0$,[8] there are explicit closed diffusion equations for the second-order scalar correlations in a random simple shear layer. As demonstrated already for the mean statistics in (13), such explicit formulas yield an explicit operator-theoretic representation for the second-order scalar statistics which can be analyzed rigorously in the limit $\bar{t}_c \to 0$.[13,14] For the moment assume that the velocity spectrum $E(k)$ in the random shear direction models from (5) and (7) is bounded and rapidly decreasing; then Kurtz's framework for the Trotter product formula[16] can be applied readily[13] in the limit $\bar{t}_c \to 0$ to yield the diffusion equation

$$\frac{\partial Q(x, t)}{\partial t} = \Delta((2\kappa + S(|x|))\, Q(x, t)), \qquad Q(x, t)|_{y=0} = Q_0(x) \tag{26}$$

In (26), $S(|x|)$ is the isotropic velocity structure function

$$\begin{aligned} S(|x|) &= \langle |v(x+x', t) - v(x, t)|^2 \rangle \\ &= 2 \fint_{S^{d-1}} \int_0^\infty [1 - \cos(2\pi x \cdot \omega \tilde{k})]\, E(\tilde{k})\, d\tilde{k}\, d\omega \end{aligned} \tag{27}$$

This argument rigorously justifies a formula of Kraichnan[17] for scalar second-order statistics with white noise correlation in time.

4.1. Inertial Range Renormalization Theory

Consider the parametrized family of spectra in (17); the inertial range consists of those scales smaller than the integral scale, defined by δ^{-1} in (17), and bigger than the dissipation scale, defined in (17) as $O(1)$ through the ultraviolet dissipation cutoff $\psi_\infty(|k|)$. One expects universal behavior for the second-order scalar statistics on scales associated with wave numbers k, satisfying $\delta \ll |k| \ll 1$ in the high-Reynolds-number limit, $\delta \to 0$.[6]

4.1.1. The Mean-Field Regime: $-\infty < \epsilon < 2$. In this regime with $-\infty < \varepsilon < 2$, the coefficient $S_\delta^\varepsilon(|x|)$ associated with $E_\delta^\varepsilon(\tilde{k})$ from (17) converges uniformly to the limiting value $S^\varepsilon(|x|)$ for $\delta = 0$; it is readily justified in this case[13,16] that $Q_\delta(x, t)$ converges to $Q(x, t)$, which satisfies the equation in (26) with coefficient $S^\varepsilon(x)$. For the inertial range renormalization theory consider the usual large-scale diffusive scaling $Q^\lambda(x, t) = \lambda^{-d} Q(x/\lambda, t/\lambda^2)$; then $Q^\lambda(x, t)$ satisfies the diffusion equation in (26) with variable coefficient $S^\varepsilon(|x|/\lambda)$. Because the limiting energy spectrum $E^\varepsilon(\tilde{k}) = V^2 |k|^{1-\varepsilon} \psi_\infty(|k|)$ is integrable for ε with $-\infty < \varepsilon < 2$, by the Riemann Lebesgue Lemma, $S^\varepsilon(|x|/\lambda)$ converges to $2\langle |v^\varepsilon|^2(0)\rangle$; thus, as $\lambda \to 0$, Q^λ converges to the Gaussian fundamental solution of the heat equation with the diffusion coefficient $2\kappa + 2\langle |v^\varepsilon|^2(0)\rangle$. Thus, in the mean-field regime for ε with $-\infty < \varepsilon < 2$, the large-scale renormalized behavior is universal and Gaussian.

4.1.2. The Anomalous Regime: $2 < \epsilon < 4$. In the anomalous regime, I concentrate on an extremely special but instructive case with $\kappa = 0$ and $\psi_\infty(|k|) \equiv 1$ and an isotropic initial distribution for the second-order scalar correlations $Q_0(|x|, t)$. There are new phenomena in this regime involving the finite effects of $\kappa \neq 0$ and $\psi_\infty(|k|) \not\equiv 1$ which are subtle and described elsewhere.[13] With all these assumptions, in the high-Reynolds-number limit, for ε with $2 < \varepsilon < 4$, it follows from (17) and (27) that $S_\delta^\varepsilon(|x|)$ converges to $D_\varepsilon |x|^{\varepsilon-2}$ as $\delta \to 0$; furthermore, it can be justified rigorously[13,16] that $Q_\delta(|x|, t)$ converges in the high-Reynolds-number limit to $Q(|x|, t)$ which satisfies the degenerate forward diffusion equation

$$\frac{\partial Q(|x|, t)}{\partial t} = D_\varepsilon \Delta(|x|^{\varepsilon-2} Q(|x|, t)), \qquad Q(|x|, t)|_{t=0} = Q_0(|x|) \tag{28}$$

Note that $\hat{Q}(|x|, t) = |x|^{\varepsilon-2} Q(|x|, t)$ formally satisfies the adjoint backward equation related to (28). In radial coordinates this backward diffusion equation is given by

$$\frac{\partial Q^*(r, t)}{\partial t} = D_\varepsilon r^{\varepsilon-1-d} \frac{d}{dr}\left(r^{d-1} \frac{d}{dr} Q^*(r, t)\right), \qquad r > 0, \quad t > 0 \tag{29}$$

It is readily checked that for the operator in (29), $r = 0$ is an entrance boundary and $r = +\infty$ is an exit boundary for $2 < \varepsilon < 4$ according to Feller's classification of singular diffusions;[18] furthermore, the coefficients are non-exploding at $r = \infty$ for $2 < \varepsilon < 4$.[16] Thus, Eq. (29) has solutions that exist and vanish rapidly for any rapidly decreasing smooth initial data; furthermore, the solutions of (29) form a strongly continuous contraction semigroup on $C_0([0, \infty))$, the continuous functions vanishing at infinity, with a natural domain of definition for the generator.[16,18] Also, the reader can verify that (28) has the explicit radial similarity solution for $2 < \varepsilon < 4$,

$$\bar{Q}_\varepsilon(r, t) = (D_\varepsilon t)^{-d/(4-\varepsilon)} F_\varepsilon\left(\frac{r}{(D_\varepsilon t)^{1/(4-\varepsilon)}}\right) \tag{30}$$

with

$$F_\varepsilon(r) = C_\varepsilon r^{2-\varepsilon} \exp\left(\frac{-r^{4-\varepsilon}}{(4-\varepsilon)^2}\right)$$

and C_ε a normalizing constant chosen so that $1 = C_\varepsilon A_{d-1} \int_0^\infty r^{d-1} F(r)\, dr$. Note that in the limit as $\varepsilon \downarrow 2$, this solution reduces to a Gaussian distribution as in the mean-field regime.

With the preliminaries in (28)–(30), finally I discuss briefly the universal inertial range renormalization theory. Consider the anomalous scaling for $2 < \varepsilon < 4$,

$$Q^\lambda(|x|, t) = \lambda^{-d} Q\left(\frac{|x|}{\lambda}, \frac{t}{\lambda^{4-\varepsilon}}\right) \tag{31}$$

I claim that in the inertial range scaling limit $\lambda \to 0$ for any $t > 0$

$$Q^\lambda(|x|, t) \quad \text{converges to} \quad \bar{Q}_\varepsilon(|x|, t) \tag{32}$$

where the convergence is weak convergence of measures in R^d. Thus, the universal probability distribution characterizing the renormalized second-order scalar statistics is the non-Gaussian distribution in (30) in the anomalous regime $2 < \varepsilon < 4$. The proof of (32) involves an adjoint integration by parts argument utilizing the backward equation[13] and careful use of two facts: (1) 0 is an entrance boundary in Feller's classification, so certain functions and appropriate normal derivatives *a priori*[18] have a well-defined limit as $r \to 0$ for functions in the domain of the generator for (29); (2) solutions of (28) inherit similar properties as in fact 1 because $\hat{Q}(r, t) = r^{\varepsilon-2} Q(r, t)$ satisfies (29) if $Q(r, t)$ satisfies (28). These two facts are crucial in treating this singular diffusion problem and are the source of new phenomena for $\kappa > 0$ and $\psi_\infty(|k|) \not\equiv 1$ in the general case. The detailed argument and generalizations are presented elsewhere.[13,14]

5. CONCLUDING REMARKS

Equation (28) together with the fundamental solution in (30) provide a rigorous derivation of properties of equations for the second-order scalar correlations initially suggested by Richardson;[19] recently, in other work Kraichnan[20] has noted the same equation as (28) for second-order scalar correlations. It is also worth mentioning that in the anomalous regime, the universal behavior of the second-order correlations expressed by $\bar{Q}_\varepsilon$ from (30) is invariant under the anomalous scaling group $(x, t) \to (x/\lambda, t/\lambda^{4-\varepsilon})$, while the equation for the mean statistics from (25) is not scale invariant in the anomalous regime $2 < \varepsilon < 4$. Thus, the renormalized mean statistics are not scale invariant, but the higher-order statistics are automatically scale invariant with anomalous scaling. This renormalized behavior is to be expected and has been demonstrated rigorously in two other simplified anisotropic models[6,8] besides the isotropic model discussed here. Unfortunately, the lack of scale invariance for the mean statistics has led some authors[21] to claim, in contradiction to the rigorous renormalization for higher-order statistics,[6,8] that these problems are not renormalizable; those authors[21] have artificially restored scale invariance for the mean statistics through nonphysical time-dependent cutoffs. These matters are discussed in detail in refs. 6 and 8.

ACKNOWLEDGMENTS

I thank Bob Kraichnan for his constructive comments. This research is partially supported by grants NSF DMS-9930194, ONR N00014-89-J-1044.P00003, ARO DAAL03-92-G-0010, and DARPA N00014-92-J-1796.

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Communicated by J. L. Lebowitz